Econometrics II

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Lecture Structure

1. Recap from last lecture
2. Projection Matrix
3. Estimation of σ^2
4. Gauss-Markov Theorem: OLS is BLUE
5. GM3 contemporaneously uncorrelated
6. GM5 Normality assumption
7. t-tests

Recap from Last Lecture I

- GM Assumptions:
  1. The true model is linear in parameters: $y = \boldsymbol{X}\beta + \varepsilon$
  2. No Perfect Collinearity: the matrix $\boldsymbol{X}$ has rank k
  3. Zero Conditional Mean: $E(\varepsilon|\boldsymbol{X}) = 0$
  4. $Var(\varepsilon|\boldsymbol{X}) = \sigma^2 I$

Recap from Last Lecture II

- Under GM1-GM3 the OLS estimator is unbiased

- Additionally imposing GM4 we can show that

$$Var(\hat{\beta}|\boldsymbol{X}) = \sigma^2(\boldsymbol{X}'\boldsymbol{X})^{-1}$$

The Projection Matrix

- In the following, we will introduce the projection matrix
- This matrix is useful for many derivations

- This matrix has dimensions $N \times N$ and is a "residual maker:" if you premultiply y with $\boldsymbol{M_X}$ you get the OLS residuals

Properties of the Projection Matrix

1. Symmetric:
2. Idempotent: $\boldsymbol{M}_{\boldsymbol{X}}^{q} = \boldsymbol{M}_{\boldsymbol{X}}$

   The last equality follows because $\boldsymbol{M}_{\boldsymbol{X}}\boldsymbol{X} = \mathbf{0}$ (see property 3)
3. $\boldsymbol{M}_{\boldsymbol{X}}\boldsymbol{X} = \mathbf{0}$
4. $\boldsymbol{M}_{\boldsymbol{X}}\hat{\varepsilon} = \hat{\varepsilon}$

Estimation of σ^2

- The variance-covariance matrix $\sigma^2(\boldsymbol{X}'\boldsymbol{X})^{-1}$ involves the disturbance variance σ^2 which is unknown
- It is reasonable to base an estimate on the RSS from the fitted regression
- We can use the projection matrix to derive this estimator:

- The last equality follows since $\boldsymbol{M_X X} = \mathbf{0}$ (see property 3 on the previous slide)
- Thus:

- The last equality follows from property 2 on the previous slide

Estimation of σ^2

- (Remember in general $tr(A) = tr \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = a_{11} + a_{22} + a_{33}$)
- Now we use two facts about traces: 1) $tr(a) = a$ if a is a scalar, 2) $tr(AB) = tr(BA)$
- From this it follows that

Estimation of σ^2

- From this it follows that:

  is an unbiased estimator of σ^2
- This is the matrix equivalent of the formula of the first weeks of the semester:
- Hence the estimated standard error of $\hat{\beta}$ is: $\hat{\sigma}^2(\boldsymbol{X}'\boldsymbol{X})^{-1}$

Gauss-Markov Theorem

- The Gauss Markov Theorem states that under GM1-4 the OLS estimator is the best linear unbiased estimator (BLUE)
- We have shown before that under GM1-3 OLS is a linear unbiased estimator (LUE)
- We now show that it is the best, i.e. most efficient estimator
- It is only the best if $\sigma^2(\boldsymbol{X}'\boldsymbol{X})^{-1}$ is a smaller variance than the variance of alternative linear estimators

Is There a Linear Estimator With Smaller Variance?

- Any linear estimator will be

- If $\boldsymbol{A}(\boldsymbol{X})$ is unbiased it must be that $\boldsymbol{A}(\boldsymbol{X})\boldsymbol{X} = \boldsymbol{I}$
- The variance of the alternative estimator would be

- Using GM4 this simplifies to:

Is There a Linear Estimator With Smaller Variance?

- If the alternative estimator had smaller variance we would have:

- Now using the fact that $\boldsymbol{A}(\boldsymbol{X})\boldsymbol{X} = \boldsymbol{I}$ we can rewrite this as

- Because M_X is symmetric and idempotent $\boldsymbol{A M_X A'}$ is positive semidefinite
- Hence, OLS is BLUE
- Note: Wooldridge also shows this proof without matrix algebra in section 3A.6. It is much more cumbersome without matrix algebra

GM Assumption 3 - Contemporaneously Uncorrelated

- Last week we showed that GM3: $E(\varepsilon|\boldsymbol{X}) = 0$ ensures that OLS is an unbiased estimator
- What happens if we assume a weaker version of GM3?
- In particular, consider GM3-contemporraneously uncorrelated (cu)

$$E(\varepsilon_i x_{ij}) = corr(\varepsilon_i, x_{ij}) = 0$$

- for all i and j

GM3cu - Unbiased?

- Would OLS remain unbiased under this weaker GM3cu?
- From before we know that:
$$\hat{\beta} = \beta + (\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\varepsilon$$
- If we take expectations we get:
$$E[\hat{\beta}] = \beta + E[(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\varepsilon]$$
- $(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'$ is a function of all x_{ij} and not just a function of a single i hence $E[(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\varepsilon] \neq 0$
- OLS would be biased in this case
- However, it can be shown that under this weaker GM3cu $\hat{\beta}$ is "aymptotically unbiased" i.e. consistent as $N \to \infty$
- The proof of this is not trivial (wait for an MSc level econometrics course)

Normality Assumption

- We add the final classical linear model assumption: $\hat{\beta}$ has a multivariate normal distribution:

- This implies:

- This assumptions allows us to carry out hypothesis tests and construct confidence intervals
- While this is a strong assumption, it can be shown that if $N \to \infty$ the distribution of the error term will converge to a Normal (proof not done in this course)

t-Tests

- Assumption GM5 also implies that:

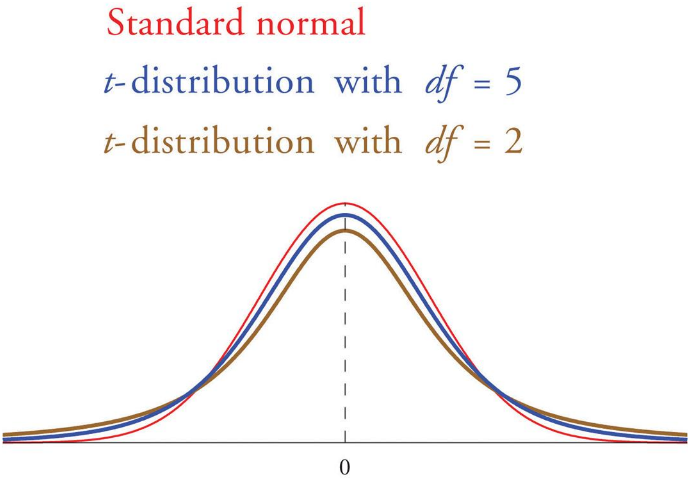
$$\hat{\beta} - \beta \sim N(0, \sigma^2(\boldsymbol{X}'\boldsymbol{X})^{-1})$$

- In practice we do not know σ^2 but can estimate $\hat{\sigma}^2$
- This, however, messes up the normality assumption:

$$\hat{\beta} - \beta \nsim N(0, \hat{\sigma}^2(\boldsymbol{X}'\boldsymbol{X})^{-1})$$

- What it the distribution in that case? We can show that it is distributed as a t-distribution and hence we can use t-tests to test hypotheses about β

t-Distribution vs. Normal Distribution



t-Distribution

- In general, suppose you have two *independent* random variables u and w with the following properties:

- Then:

χ^2-Distribution

- In general, if we sum the squares of N independent standard normal random variables then this sum is distributed as a chi-square distribution:
- e.g. if $v \sim N(0, I_N)$ then:

$$v'v \sim \chi^2(N)$$

- If $v \sim N(0, I_N)$ and $\boldsymbol{A}$ is an idempotent matrix with $rank(\boldsymbol{A}) = q$ then:

$$v'\boldsymbol{A}v \sim \chi^2(q)$$

Distribution of the Test Statistic

- Now we show that the standard test statistic for t-test is distributed as a t-distribution
- Under GM5:
- Hence:
- From above we also know:

Distribution of the Test Statistic

- Now we define the test statistic as follows

- The first equality follows because $\boldsymbol{M_x}\varepsilon = \boldsymbol{M_x}(y - \boldsymbol{X}\beta) = \boldsymbol{M_x}y - \boldsymbol{M_x}\boldsymbol{X}\beta = \boldsymbol{M_x}y = \hat{\varepsilon}$, hence $\varepsilon'\boldsymbol{M_x}\varepsilon = \varepsilon'\boldsymbol{M_x'}\boldsymbol{M_x}\varepsilon = \hat{\varepsilon}'\hat{\varepsilon}$
- We can rewrite this as:

- This is the standard formula for the t-test: $\frac{\hat{\beta}-\beta^0}{se(\hat{\beta})}$